Incremental variational principles with application to the homogenization of nonlinear dissipative composites

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INCREMENTAL VARIATIONAL PRINCIPLES WITH APPLICATION TO THE HOMOGENIZATION OF NONLINEAR DISSIPATIVE COMPOSITES

Pierre Suquet*, Noel Lahellec *,**
* *Mechanics and Acoustics Laboratory, CNRS, Marseille, France*
** *Aix-Marseille University*

Summary This study is devoted to the overall response of nonlinear composites composed of phases which have a partly reversible and partly irreversible behavior, typically elasto-viscoplastic constituents. Upon use of an implicit time-discretization scheme, the evolution equations describing the constitutive behavior of the phases are reduced to the minimization of an incremental energy function. This minimization problem is rigorously equivalent to a nonlinear thermoelastic problem with a nonuniform transformation field. Two different techniques for approximating the nonuniform eigenstrains by piecewise uniform eigenstrains and for linearizing the nonlinear thermoelastic problem will be presented.

INCREMENTAL VARIATIONAL PRINCIPLES FOR DISSIPATIVE MATERIALS

Individual constituents

The composite materials considered in this study are composed of individual constituents exhibiting a dissipative behavior which can be modelled in the general framework of constitutive relations deriving from two thermodynamic potentials. The specification of the constitutive relations requires first the identification of a finite number of *internal variables* $\boldsymbol{\alpha}$ in addition to the observable strain $\boldsymbol{\varepsilon}$ (attention is limited here to infinitesimal strains). The corresponding *driving forces* $\boldsymbol{\mathcal{A}}$ and the stress $\boldsymbol{\sigma}$are derived, as usual, from the *free-energy function* $w(\boldsymbol{\varepsilon},\boldsymbol{\alpha})$ of the material which encompasses all the energy available to trigger the material evolution:

$$\boldsymbol{\sigma}=\frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\boldsymbol{\alpha}),\quad \boldsymbol{\mathcal{A}}=-\frac{\partial w}{\partial \boldsymbol{\alpha}}(\boldsymbol{\varepsilon},\boldsymbol{\alpha}).$$

These driving forces govern the evolution of the internal variables $\boldsymbol{\alpha}$ by derivation of the dissipation potential $\varphi(\dot{\boldsymbol{\alpha}})$

$$\boldsymbol{\mathcal{A}}=\frac{\partial \varphi}{\partial \dot{\boldsymbol{\alpha}}}(\dot{\boldsymbol{\alpha}}),\quad \text{or equivalently}\quad \dot{\boldsymbol{\alpha}}=\frac{\partial \varphi^*}{\partial \boldsymbol{\mathcal{A}}}(\boldsymbol{\mathcal{A}}),\quad \text{where } \varphi^* \text{ is the Legendre transform of } \varphi,$$

Upon elimination of $\boldsymbol{\mathcal{A}}$, it is seen that the history of the internal variables is obtained by solving a differential equation in time, whereas the strain, which appears as a forcing term in this differential equation, is related to the stress by derivation of the free-energy of the system

$$\frac{\partial w}{\partial \boldsymbol{\alpha}}(\boldsymbol{\varepsilon},\boldsymbol{\alpha})+\frac{\partial \varphi}{\partial \dot{\boldsymbol{\alpha}}}(\dot{\boldsymbol{\alpha}})=0,\quad \boldsymbol{\sigma}=\frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\boldsymbol{\alpha}),$$

In the present study, the two potentials w and φ are convex functions of their arguments.

Incremental potential

The time interval of study $[0,T]$ is discretized into sub-intervals $t_0=0,t_1,....,t_n,t_{n+1},...,t_N=T$, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ denote the stress and strain field at at time t_{n+1} and $\boldsymbol{\alpha}_n$ denotes the state variables at the previous time step. After time-discretization the integration of the differential equation for the internal variables $\boldsymbol{\alpha}$ reduces to the minimization with respect to all possible internal variables of an *incremental potential* J, which defines in turn the condensed potential w_Δ from which the stress can be obtained by derivation with respect to the strain

$$\boldsymbol{\sigma}=\frac{\partial w_\Delta}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}),\quad w_\Delta(\boldsymbol{\varepsilon})=\inf_{\boldsymbol{\alpha}}\quad J(\boldsymbol{\varepsilon},\boldsymbol{\alpha}),\quad J(\boldsymbol{\varepsilon},\boldsymbol{\alpha})=w(\boldsymbol{\varepsilon},\boldsymbol{\alpha})\ +\ \Delta t\ \varphi\left(\frac{\boldsymbol{\alpha}-\boldsymbol{\alpha}_n}{\Delta t}\right).$$

COMPOSITE MATERIALS

Reduction to a single potential

A representative volume element (r.v.e.) V of the composite consisting of N different phases occupying domains $V^{(r)}$ with volume-fraction $c^{(r)}$. The thermodynamic potentials for phase r are denoted by $w^{(r)}$ and $\varphi^{(r)}$. The r.v.e. V is subjected to a path of macroscopic strain $\boldsymbol{E}(t)$ and the local fields of stress, strain, and internal variables are denoted by $\boldsymbol{\sigma}(\boldsymbol{x},t)$, $\boldsymbol{\varepsilon}(\boldsymbol{x},t)$ and $\boldsymbol{\alpha}(\boldsymbol{x},t)$ respectively. The effective response of the composite is the history of the overall stress $\boldsymbol{\Sigma}(t)$, which is the volume average of the local stress field over the r.v.e., as a function of the overall strain $\boldsymbol{E}(t)$.

After time-discretization, the determination of the local fields in the composite at time t_{n+1} is reduced to the determination of the effective energy $\tilde{w}_\Delta$ of a composite material with *one potential* w_Δ in the r.v.e.

$$\boldsymbol{\Sigma}=\frac{\partial \widetilde{w}_\Delta}{\partial \boldsymbol{E}}(\boldsymbol{E}),\quad \tilde{w}_\Delta\left(\boldsymbol{E}\right)=\inf_{\langle\boldsymbol{\varepsilon}\rangle=\boldsymbol{E}}\ \langle w_\Delta(\boldsymbol{\varepsilon})\rangle=\inf_{\langle\boldsymbol{\varepsilon}\rangle=\boldsymbol{E}}\ \left\langle \inf_{\boldsymbol{\alpha}}\quad J(\boldsymbol{\varepsilon},\boldsymbol{\alpha})\right\rangle.$$

Therefore the homogenization problem for composite materials with two potentials is reduced to a homogenization problem for materials with a *single* potential. There are however two important features of this variational problem which should be highlighted.

1. First, the local condensed potential w_Δ is given as the minimum of the full incremental potential J with respect to the internal variables and is *not explicitly known*. Therefore it is easier to work with the full incremental potential J rather than with the condensed potential itself. In addition J is nonquadratic and therefore it has to be handled using a *nonlinear* homogenization theory. A linearization is needed in order to render the problem amenable to analytical calculations. Several possible linearizations are possible (Ponte Castañeda and Suquet, 1998) and two of them, the isotropic secant method and a new anisotropic second-order method (Lahellec and Suquet, 2007), have been implemented.

2. Second, the condensed potential w_Δ depends on $\boldsymbol{x}$ not only through the characteristic functions $\chi^{(r)}(\boldsymbol{x})$ of the individual phases, but also through the fields $\boldsymbol{\alpha}(\boldsymbol{x})$ and $\boldsymbol{\alpha}_n(\boldsymbol{x})$. In other words $w_\Delta(\boldsymbol{\varepsilon})$ *is not a uniformly defined function over phase* r.

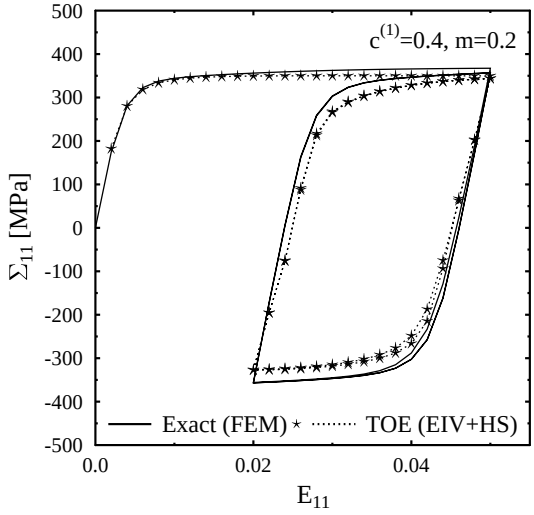
Effective internal variables (EIV)

This second difficulty is closely related to the intra-phase heterogeneity of the field of internal variables (typically the plastic strain field). This difficulty can be overcome (in an approximate manner) by replacing the field $\boldsymbol{\alpha}_n(\boldsymbol{x})$ by an *effective internal variable* $\boldsymbol{\alpha}_n^{(r)}$ which is a single tensor (instead of a field) representative of the whole field of internal variables over the entire phase r.

A first natural guess for the effective internal variable $\boldsymbol{\alpha}_n^{(r)}$ would be to consider the fields $\boldsymbol{\alpha}_n(\boldsymbol{x})$ and $\boldsymbol{\alpha}(\boldsymbol{x})$ to be piecewise uniform within each individual phase. This choice leads rigorously to the Transformation Field Analysis (TFA) which overlooks the heterogeneitty of the plastic strain field and is known to yield too stiff predictions. Other propositions have been made (Lahellec and Suquet, 2007) making use of the first and the second moments of the internal variables in each individual phases and leading to more accurate predictions.

Discussion

In order to assess the accuracy of the present model, its predictions have been compared to numerical simulations of the response of a fiber-reinforced composite. Linear-elastic fibers are aligned in the third direction with characteristics $E^{(1)} = 288889\,\mathrm{MPa}$, $\nu^{(1)} = 0.3$. The matrix is elastoviscoplastic with the following constitutive relations



$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{\mathsf{e}} + \dot{\boldsymbol{\varepsilon}}^{\mathsf{vp}}, \quad \dot{\boldsymbol{\varepsilon}}^{\mathsf{vp}} = \frac{3}{2}\dot{\varepsilon}^{\mathsf{vp}}_{\mathsf{eq}}\frac{\boldsymbol{s}}{\sigma_{\mathsf{eq}}}, \quad \dot{\varepsilon}^{\mathsf{vp}}_{\mathsf{eq}} = \dot{\varepsilon}_0\left(\frac{\sigma_{\mathsf{eq}}}{\sigma_0}\right)^{1/m},$$

and material characteristics

$$E^{(2)} = 70000\,\mathrm{MPa},\ \nu^{(2)} = 0.3,\ \dot{\varepsilon}_0 = 1,\ \sigma_0 = 480\,\mathrm{MPa}.$$

The two thermodynamic potentials w and φ for the fibers and the matrix are straightforward. The applied loading is an in-plane shear

$$\boldsymbol{E} = E_{11}(t)\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2)\,.$$

The loading consists of 3 different regimes: loading, partial unloading and reloading.

The model is implemented using the (anisotropic) linearization proposed by Lahellec and Suquet (2007). The problem is reduced to a thermoelasticity problem for an anisotropic linear composite. The effective energy of this comparison solid is determined by the Hashin-Shtrikman (HS) theory. As can be seen from the figure, the model can handle both loading and unloading conditions and its predictions compare well with the full-field simulations.

References

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